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LOCATING ABSOLUTE 2-CENTERS OF UNDIRECTED GRAPHS

by

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UNITED STATES
NAVAL POSTGRADUATE SCHOOL

THESIS

LOCATING ABSOLUTE 2-CENTERS
OF
UNDIRECTED GRAPHS

by

Clarke McKinley Gillespie, Jr.

DECEMBER 1968

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LOCATING ABSOLUTE 2-CENTERS
OF
UNDIRECTED GRAPHS

by

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ABSTRACT

This study analyzes the location of vertex and absolute 2-centers of an undirected graph. Under certain assumptions, these locations would be useful for determining the optimal positioning of emergency facilities such as fire stations. Vertex and absolute multi-centers are defined, and a procedure for locating the vertex multi-centers is given. It is shown that certain combinations of arcs and vertices never contain absolute 2-centers, while certain others will always contain a 2-center which is more centrally located than the vertex 2-center. Although no algorithm was found for determining the absolute 2-center of a graph, an algorithm is presented for finding the best 2-centers which exist on arcs incident to the vertex 2-center.

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CHAPTER 1

INTRODUCTION

Graph theoretic concepts may be usefully applied to the problem of optimal location of emergency facilities, such as fire stations or repair shops. The area to be covered by the emergency equipment can be modeled by a graph $G(V, A)$, where V is the set of vertices, which corresponds to the various localities to be served, and A is the set of arcs, which corresponds to the roadways interconnecting these locations.

The problem of optimal distribution of facilities arises in determining the best location for these emergency facilities within the area to be covered. Since minimum response time to a call for assistance is the objective, and since this time is dependent on the distance the emergency equipment must travel, it is desirable to place the facilities in such a position that all the localities are as near as possible to one of the facilities. The problem then is to optimally locate available equipment. The solution to the problem lies in determining the location of multi-centers of the graph.

The optimum location problem has been examined in many contexts by various investigators, going back as far as the 17th century. For an extensive bibliography on the subject, see Francis.[1]

The research reported herein is based largely on work reported in two articles by Hakimi.[2, 3] Hakimi presents a solution technique for optimally locating a single center on a graph. He also poses the problem of locating multi-centers, but has no solution procedure.

This report discusses a solution procedure for optimally locating 2-centers. The technique is basically an extension of Hakimi's technique for the single center. The technique will find optimal 2-center locations when they occur on certain arcs. Optimal 2-centers can occur on other arcs. The same basic technique can be used to determine these others, but only with a great deal of effort.

Chapter 2 introduces the graph theoretic concepts appropriate to the discussion and summarizes Hakimi's solution to the single-center case. Chapter 3 is devoted to a series of theorems whose net result is to alleviate somewhat the work required to locate 2-centers. Chapter 4 presents the solution technique, with an illustrative example. Finally, Chapter 5 summarizes the report.

CHAPTER 2

LOCATION OF THE ABSOLUTE CENTER

A few concepts from classical graph theory are an appropriate preliminary to a discussion of Hakimi's paper. A graph $G(V, A)$ consists of a set of vertices (or nodes) V, connected by a set of arcs, A. Vertices will be denoted by letters or numbers in parentheses, e.g., (p), (q), (3). A general point on the graph, not necessarily at a vertex, will be indicated by the letter x. The letter v will indicate any fixed but unspecified vertex.

Arcs will be denoted by pairs of letters or numbers in parentheses signifying the vertices joined by the arc, i.e., (p, q) is the arc joining vertex p to vertex q. In this paper all arcs are undirected, i.e., (p, q) is equivalent to (q, p).

Attached to each arc of a graph is a non-negative number signifying its length (or cost to traverse it or similar measure). Similar numbers may be attached to vertices, although none will be in this discussion. Thus, the distance between any two vertices is the sum of the lengths of the arcs between them. In general, there will be more than one path between vertices.

If we define

$$d(x, y) \equiv \text{minimum distance from } (x) \text{ to } (y),$$

then for any vertex c, there will exist $d(c, v)$ for all $v \in V$. The distance to the vertex farthest from (c) is called the radius associated with the vertex c, or

$$r(c) \equiv \max_{v \in V} d(c, v) \text{ for a fixed vertex } c.$$

The minimum of the radii associated with all vertices of a graph is the radius of the graph, or

$$r_0 = r(v^*) = \min_{c \in V} r(c). \qquad [1]$$

The vertex v^* satisfying equation [1] has been defined as the center of the graph in the classical literature.[4, 5] This definition of center for a graph has some intuitive appeal since v^* is in some sense "closer" to the vertices than any other vertex.

Hakimi shows that points that are more centrally located than the center of the graph may exist along arcs. The radius associated with such points would then be less than r_0. He refers to the point which yields the minimum radius as the absolute center. More precisely, a point x^* on the graph G is called an absolute center if

$$\max_{v \in V} d(v, x^*) \leq \max_{v \in V} d(v, x; x \text{ on } G).$$

Hakimi's approach for determining the absolute center is to find on each arc the point which yields the minimum radius. These points he calls local centers. The local center with the smallest minimum radius is the absolute center of the graph.

Without going into the details of Hakimi's method, it is worthwhile to look at his method for determining the location of local centers. Consider the generalized graph of Figure 1 (by "generalized" we mean it may have any number of vertices with any connectivity between them). Let B be the length of the arc (p, q) and x be the distance from (p) to point x. It follows that

$$d(x, i) = \min [x + d(p, i), \; B - x + d(q, i)].$$

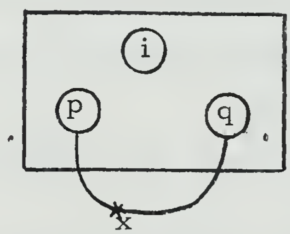

Figure 1. An illustration of the distance $d(x, i)$

Let $f(x) = x + d(p, i)$ and $g(x) = B - x + d(q, i)$. The functions $f(x)$ and $g(x)$ are plotted in Figure 2. The distance $d(x, i)$ is shown by the heavy lines of the graph. For every vertex i in the graph similar plots can be obtained. Then in Figure 3, all of these plots

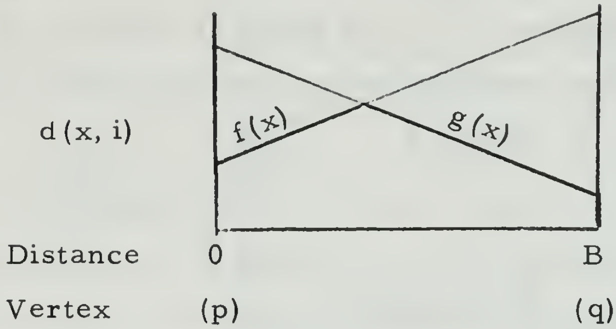

Figure 2. A plot of d(x, i) for arc (p, q)

are next combined and shown on the same axes. The curve of the function

$$F(x) = \max_{i \in V} d(x, i)$$

for arc (p, q) is then found by selecting the maximum value of all the functions plotted for each value of x (the heavy line in Figure 3).

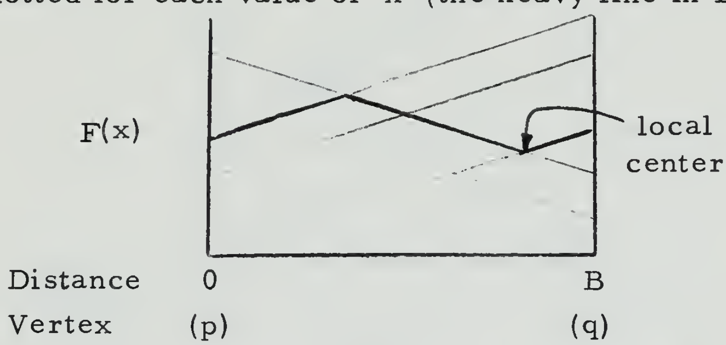

Figure 3. A plot of F(x) for arc (p, q)

The local center of arc (p, q) is the value of x associated with the minimum value of F(x). Finally, the absolute center is that local center having the smallest minimum value of F(x) of all local centers.

The simple example shown in Figure 4 will serve to illustrate Hakimi's method. In Figure 5, the plots for each arc are shown. The

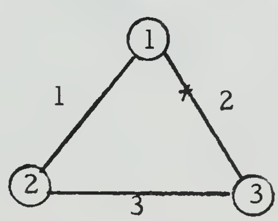

Figure 4. Example graph

center of the graph (in the classical sense) is (1), with $r_0 = 2$. The absolute center is located 1/2 unit from (1), along (1, 3), and has a radius of 1 1/2. The absolute center determined from Figure 5b is indicated by the x in Figure 4.

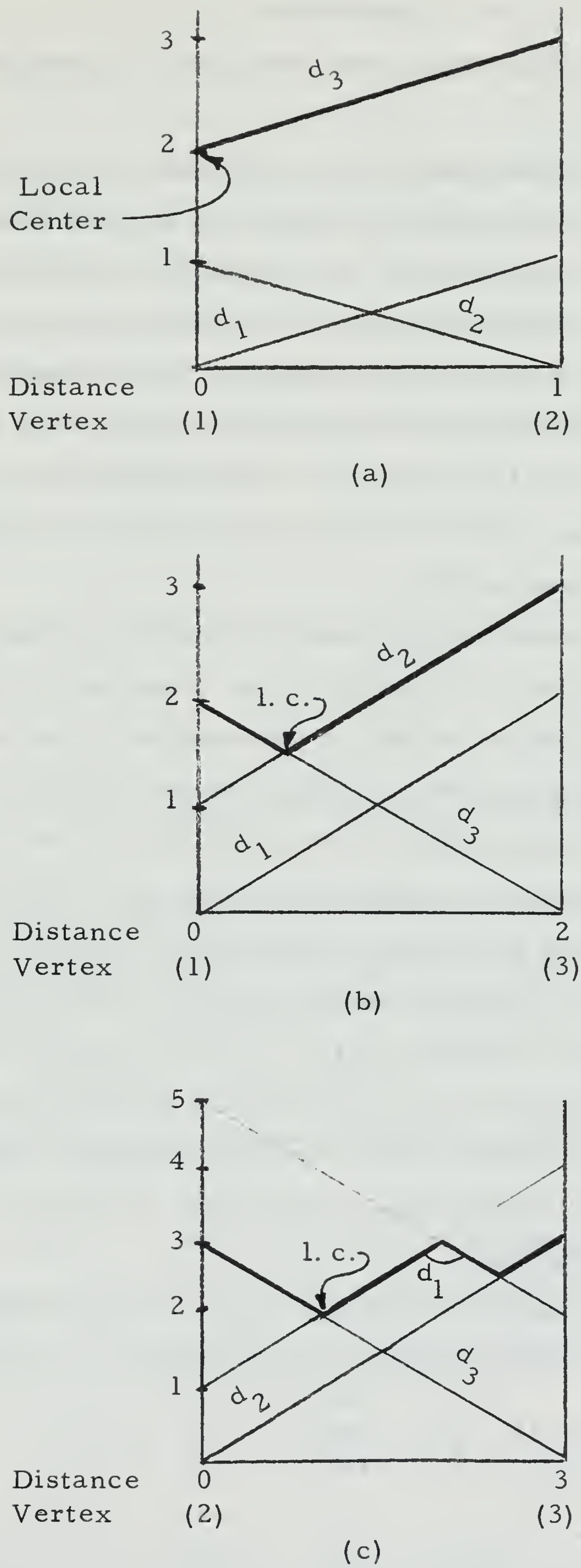

Figure 5. Determination of local centers for example graph.

CHAPTER 3

VERTEX 2-CENTERS AND ABSOLUTE 2-CENTERS

The purpose of this chapter is to introduce the concept of the vertex 2-center and the absolute 2-center and explore some of their characteristics. Specifically, as a valuable part of the solution technique, it will be shown that certain combinations of arcs and vertices can never constitute an absolute 2-center. Consequently, by reducing the number of arcs to be examined in the search for a solution, a decrease in solution time results. It will also be shown that under certain conditions, a solution that improves on the vertex 2-center can be obtained immediately.

Suppose that several towns and the highways joining them can be reasonably modeled by a graph $G(V, A)$. Suppose further that there are two fire stations to be built to provide protection for all the towns. The problem then is to select the two towns in which to build the firehouses, assuming initially that the firehouses must be built in towns.* Since the two firehouses are to service all the towns, and since fast response to an alarm is desired, the towns should be selected so that the farthest distance any fire truck will have to go is the minimum for all possible pairs of towns considered. The problem as treated here assumes that only one fire can be burning at a time, and that when one station answers a call, the other station takes no action. Selection of the towns corresponds to finding a vertex 2-center of the associated graph.

In general, a set of p vertices, V_p^*, on G is called a vertex p-center of G, if for every set of p vertices V_p on G,

$$\max_{i \in V} d(i, V_p) \geq \max_{i \in V} d(i, V_p^*) = r_p ,$$

---

* Building both in one town will never be better than selecting two towns, although it may be as good.

where r_p is defined as the radius associated with the vertex p-center. In this paper, only the case where $p = 2$ will be considered.

To find a vertex 2-center, the simplest method is to construct a matrix $D = \{d_{ij}\}$, where $d_{ij} = d(i, j)$, the shortest distance from vertex i to vertex j. For an undirected graph, D will be a symmetric matrix. If there are n vertices, there will be $\binom{n}{2}$ sets of two vertices to be examined. The set yielding the minimum radius will be the vertex 2-center.

Suppose, for example, that a graph has the following D matrix:

$$D = \begin{bmatrix} 0 & 2 & 5 & 3 & 5 & 4 \\ 2 & 0 & 3 & 4 & 5 & 6 \\ 5 & 3 & 0 & 3 & 2 & 4 \\ 3 & 4 & 3 & 0 & 5 & 7 \\ 5 & 5 & 2 & 5 & 0 & 2 \\ 4 & 6 & 4 & 7 & 2 & 0 \end{bmatrix}$$

Since there are six vertices, there are $\binom{6}{2} = 15$ possible vertex 2-centers.

The first step in determining a vertex 2-center is to determine the minimum distance from each pair of vertices to each other vertex. For instance, considering $V_2 = \{(1), (2)\}$,

$$d\{(3), V_2\} = \min(5, 3) = 3.$$

Similarly, the minimum distance to all other vertices can be determined.

The next step is to determine the radius associated with each V_2. The radius associated with a pair of vertices is the distance to the farthest vertex, or the maximum of these minimum distances. Thus, the radius associated with $V_2 = \{(1), (2)\}$ is

$$\max_v d(v, V_2) = \max_v (0, 0, 3, 3, 5, 4) = 5.$$

The final step is to select a vertex 2-center, a V_2 which has the minimum radius. Thus, the radius of the graph for the 2-center case is $r_2 = \min_{V_2 \subset V} [\max_v d(v, V_2)]$, and the V_2 which produces r_2 is V_2^*, a vertex 2-center. For the example given, the reader may verify that $V_2^* = \{(1), (5)\}$, with $r_2 = 3$.

The last expression can be generalized as an alternate definition of the radius associated with a vertex p-center, that is,

$$r_p = \min_{V_p \subset V} [\max_v d(v, V_p)],$$

and the V_p which produces r_p is V_p^*.

Examining all possible sets of vertices would be a sizeable job for a large graph, but could be accomplished very easily by a computer.

Now suppose that the requirement were lifted that the firehouses must be constructed in towns. Instead, they may be constructed at any two points either in towns or along the highways. The same criterion of minimum response time is, of course, appropriate.

The solution to the problem now corresponds to finding an absolute 2-center of the associated graph. For the general case, a set of p points X_p^* is called the absolute p-center of G, if for every set of p points X_p on G,

$$\max_{i \in V} d(i, X_p) \geq \max_{i \in V} d(i, X_p^*) = r'_p.$$

Again, $p = 2$ is the only case considered here.

The algorithm for finding an absolute 2-center is the subject of Chapter 4. The remainder of this chapter is concerned with conditions necessary for the existence of vertex and absolute 2-centers.

<u>Theorem 1.</u> Any graph having at least p vertices has a vertex p-center.

This theorem is true from the definition of the vertex p-center. Since for any graph with p vertices there will be at least one set V_p, then there must exist a set V_p^*, the vertex p-center.

Theorem 2. Any graph having at least p vertices has an absolute p-center.

This theorem follows directly from Theorem 1. Since V^*_p always exists, and $V^*_p \subset X_p$, then X^*_p must exist.

It should be noted that neither the vertex 2-center nor the absolute 2-center need be unique.

Lemma 1. Let $V^*_2 = \{(p), (q)\}$ be the vertex 2-center with radius r_2, and (p, q) be the arc joining (p) and (q). There may exist an interval on (p, q) such that for any point x located in that interval, the set $\{(p), x\}$ also has radius r_2.

Proof: The set V may be partitioned into two subsets, $V_p = \{v; d(v, p) \leq d(v, q)\}$ and $V_q = \{v; d(v, q) \leq d(v, p)\}$. Vertices for which $d(v, p) = d(v, q)$ may be assigned to either subset. Let s be the vertex in V_p for which $d(s, p) = \max_{v \in V_p} d(v, p)$.

Similarly, define t to be such that $d(t, q) = \max_{v \in V_q} d(v, q)$. Without loss of generality we assume

$$d(s, p) = r_2 ;$$

$$d(t, q) = r' \leq r_2 .$$

If strict inequality holds in the second relation, then we may move along (p, q) from (q) toward (p) to any point x such that

$$r' < d(t, x) \leq r_2$$

with the radius of the set $\{(p), x\}$ remaining constant at r_2. If $r' = r_2$, then no such points exist.

Example:

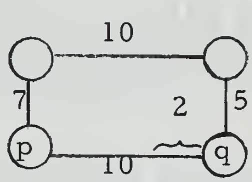

Figure 6. Example graph

For the graph of Figure 6, $r_2 = 7$ and $r' = 5$. A set consisting of (p) and any point within two units of (q) along the arc (p, q) will also have a radius of 7.

Thus, in general, it is true that we may move a distance of $r_2 - r'$ along the arc connecting the vertex 2-centers without increasing the radius r_2. Note, however, that we are not improving on r_2. In Theorem 3 and its corollaries, it will be shown that in the general case, no point on this arc will enable us to improve on r_2.

Now consider the generalized graph of Figure 7. The graph may have any number of vertices and any connectivity between the vertices. The next theorem will prove that a single arc connecting the members of a 2-center will contain no points which may be paired with either member of the 2-center to form an absolute 2-center.

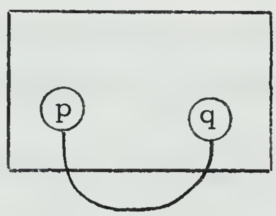

Figure 7. Generalized graph with single arc connecting the members of the 2-center.

Theorem 3. If $V_2^* = \{(p), (q)\}$ is a vertex 2-center, and $r_2 = r'$, then there exists no point x on the arc (p, q), not including (p) or (q), such that $V_2 = \{(p), x\}$ or $V_2 = \{x, (q)\}$ is an absolute 2-center.

Proof: Let b = the length of the single arc joining (p) and (q). Then

$$d(v,x) = \min\{d(v,p) + \alpha b;\ d(v,q) + (1-\alpha) b\},\quad 0 < \alpha < 1.$$

For any point x on the arc, there will exist two subsets of V which are defined as follows:

$$V'_p = \{v;\ d(v,x) = d(v,p) + \alpha b\}$$
$$V'_q = \{v;\ d(v,x) = d(v,q) + (1-\alpha) b\}$$

where $V'_p \cup V'_q = V$.

For every $v \in V'_p$, $d(v,x) > d(v,p)$ and

$$\max_{v \in V'_p} d(v, x) > \max_{v \in V'_p} d(v, p). \qquad [2]$$

Therefore, the radius associated with the point x and the set V'_p is always greater than the radius associated with (p) and V'_p. Similarly, for every $v \in V'_q$,

$$d(v, x) > d(v, q)$$

and

$$\max_{v \in V'_q} d(v, x) > \max_{v \in V'_q} d(v, q) . \qquad [3]$$

Thus the radius associated with the point x and the set V'_q is always greater than the radius associated with (q) and V'_q.

Examining the right-hand side of inequalities [2] and [3], we may write

$$\max \{ \max_{v \in V'_p} d(v, p), \max_{v \in V'_q} d(v, q) \} = r_2 \qquad [4]$$

since $V'_p \cup V'_q = V$ and $\{(p), (q)\}$ is a vertex 2-center. Equality would hold in the case where $V'_p = V_p$, $V'_q = V_q$ where V_p and V_q are defined earlier in this chapter. Therefore, we may collect the left-hand side of [2] and [3], and write

$$\max \{ \max_{v \in V'_p} d(v, x), \max_{v \in V'_q} d(v, x) \} > r_2 ,$$

proving the theorem.

In the proof of Theorem 3 the assumption was made that $r_2 = r'$. In Corollary 1 this assumption is withdrawn.

<u>Corollary 1.</u> If $V_2^* = \{(p), (q)\}$ is the vertex 2-center, and $r_2 > r'$, then there exists no point x located more than $r_2 - r'$ from (q) on the arc (p, q) such that $V_2 = \{(p), x\}$ or $V_2 = \{x, (q)\}$ is an absolute 2-center.

Proof: The proof of this theorem follows immediately from Lemma 1 and Theorem 3. In Lemma 1 it was shown that with (p) as one member of the vertex 2-center, the other member may be located anywhere in an interval of length $r_2 - r'$ from (q) on (p, q) with the radius remaining constant at r_2. At the point $r_2 - r'$ from (q) on (p, q), the radius associated with that point and V_q is r_2. The remainder of the arc is therefore precisely analogous to the total arc under the condition $r_2 = r'$, so that the results of Theorem 3 apply directly.

Corollary 2. If $V_p^* = \{(p), (q)\}$ is a vertex 2-center, then there exists no pair of points $\{x, y\}$ on the arc (p, q) such that $\{x, y\}$ is an absolute 2-center.

The proof of this corollary follows immediately from the proof of Theorem 3. Clearly, if we are unable to find a combination of a vertex and any point on the arc that will yield a radius less than r_2 we will be unable to find two points on the arc such that the radius is less than r_2.

The next step is to extend the results above from the case where (p) and (q) are connected by a single arc to the case where they are connected by a series of arcs. For the simplest case, as is shown in Figure 8, assume that there is a single vertex (r) between (p) and (q) with no direct connectivity other than to (p) and (q). Assume that the path shown represents the shortest distance between (p) and (q), i. e., $d(p, q) = d(p, r) + d(r, q)$.

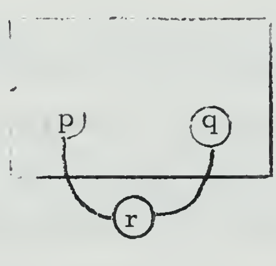

Figure 8. Generalized graph with one vertex located between members of a vertex 2-center.

Under the assumptions above, five cases may arise:

1. Both of the arcs may be less than r_2 in length,
2. One arc may be less than r_2, the other equal to r_2,

3. Both may be equal to r_2 in length,

4. One arc may be greater than r_2, the other equal to r_2, and

5. One arc may be less than r_2, the other greater than r_2.

Both arcs cannot be greater than r_2, since this would imply that the radius associated with $\{(p), (q)\}$ is greater than r_2.

Case 1: Both arcs less than r_2 in length. This is the only case in which $d(p, q)$ may be less than r_2. This fact makes discussion of this case completely analogous to the proof of Theorem 3 earlier. Stated briefly, for any point located on either arc shown in Figure 8, the set of all vertices can be partitioned into two subsets: those to which the distance is shortest through (p), and those to which the distance is shortest through (q). Of the former subset, which we will call V'_p, it is always true that $d(v, p) < d(v, x)$. Of the latter subset, call it V'_q, it is true that $d(v, q) < d(v, x)$. While it is true that the radius associated with $\{(p), (q)\}$ and one of these subsets may be less than r_2, it is also true that the maximum of the radii associated with the two subsets must be greater than (or at least equal to) r_2; otherwise, $\{(p), (q)\}$ would not be the vertex 2-center. And, since for whichever of the vertices at which this occurs the minimum distance to our point x must be larger, it must be true that no point on the pair of arcs will improve on r_2.

Thus for Case 1, both Theorem 3 and Corollaries 1 and 2 apply, i. e., there exists no point on the pair of arcs that will improve on r_2 when paired with (p) or (q), nor any pair of points which will improve on r_2.

Case 2: One arc less than r_2, the other equal to r_2. In order for this case to occur, another vertex must exist which is also located a distance r_2 from V_2^*, since $d(r, V_2^*) < r_2$, yet r_2 is the radius. This vertex will be called (m).

Theorem 4. For the conditions of this case, there exists no point x on either arc such that set $\{(p), x\}$ or $\{x, (q)\}$ is an absolute 2-center.

Proof: Let $d(p, r) = r_2$ and $d(q, r) < r_2$. Two situations can arise in this case: either $d(p, m) = r_2$ and $d(q, m) \geq r_2$, or $d(p, m) \geq r_2$ and $d(q, m) = r_2$.

First, let $d(p, m) = r_2$ and $d(q, m) \geq r_2$. For any point x on (p, r) located ϵ from (p),

$$\begin{aligned} d(x, m) &= d(p, m) + \epsilon \\ &= r_2 + \epsilon \\ \therefore d(x, m) &> r_2 . \end{aligned}$$

It would be useless to investigate points on the arc (q, r), since $d(q, m) \geq r_2$ at the outset, and movement toward (r) along (q, r) would only increase the distance to (m).

Second, let $d(p, m) \geq r_2$ and $d(q, m) = r_2$. As in the previous paragraph, any point on (p, r) or (q, r) will be greater than $d(p, m)$ or $d(q, m)$, and can never serve to reduce r_2.

Case 3: Both arcs equal to r_2 in length. In order to discuss this case, it will be convenient to introduce some additional notation.

Define

$$r'_p = \max_{\substack{v \in V_p \\ v \neq r}} d(v, p), \text{ and}$$

$$r'_q = \max_{\substack{v \in V_q \\ v \neq r}} d(v, q) ,$$

Observe that both r'_p and r'_q must be less than or equal to r_2. The physical interpretation of r'_p and r'_q is that they represent what would be r_2 and r' if (r) were removed from the graph and a single arc then connected (p) and (q). Select (p) and (q) such that $r'_p \geq r'_q$.

Define ϵ to be a distance along (p, r) measured from (p), and δ to be a distance along (q, r) measured from (q).

Theorem 5. For the conditions of this case, if $r'_p \leq \frac{1}{2}(r_2 + r'_q)$, then the minimum radius obtainable for two points on (p, r) and (q, r) is

$$r = \frac{1}{2}(r_2 + r'_q).$$

The points giving this radius occur at

$$\delta = \frac{1}{2}(r_2 - r'_q)$$

and any

$$\varepsilon \leq \frac{1}{2}(r_2 + r'_q) - r'_p .$$

If $r'_p \geq \frac{1}{2}(r_2 + r'_q)$, then the minimum radius is

$$r = r'_p .$$

The points giving this radius occur at

$$\varepsilon = 0 ,$$

and any

$$r_2 - r'_p \leq \delta \leq r'_p - r'_q .$$

Proof: In order to have a radius less than r_2, one member of a 2-center must be closer than r_2 to (r). Movement along (q, r) toward (r) gives such points, but also extends the distance to all other vertices in V_q. The minimum distance that can be obtained occurs at a distance d from (q) along (q, r) where

$$r'_q + \delta = r_2 - \delta$$

resulting in

$$\delta = \frac{1}{2}(r_2 - r'_q) .$$

This minimum distance is

$$r = \frac{1}{2}(r_2 + r'_q) .$$

For this distance to be the radius, however, the other member of the 2-center must be located such that the distance from it to the farthest vertex in V_p must be less than or equal to this distance. If this, in fact, occurs, that is, if

$$r'_p \leq \frac{1}{2}(r_2 + r'_q) ,$$

then the other member of the 2-center is free to move, so long as it doesn't extend the distance to the farthest vertex in V_p beyond r.

The value of ϵ is bounded,

$$r'_p + \epsilon \le \frac{1}{2}(r_2 + r'_q) ,$$

resulting in

$$\epsilon \le \frac{1}{2}(r_2 + r'_q) - r'_p .$$

It is possible that

$$r'_p \ge \frac{1}{2}(r_2 + r'_q) .$$

When this occurs, one member of the 2-center must remain at (p); that is, $\epsilon = 0$ in order not to extend the radius beyond r'_p. The other member is free to move, so long as the distance from it to both (r) and the farthest member of V_q never exceeds r'_p. Thus,

$$r'_q + \delta \le r'_p$$

$$\delta \le r'_p - r'_q$$

and

$$r_2 - \delta \le r'_p$$

$$\delta \ge r_2 - r'_p$$

or

$$r_2 - r'_p \le \delta \le r'_p - r'_q .$$

At first glance it may seem that we should consider points on (p, r). Note that in general the minimum radius obtainable for points on (p, r) is

$$\max \left[\frac{1}{2}(r_2 + r'_p),\ r'_q \right] .$$

However, since r'_p was selected greater than r'_q, the radius is in fact always $\frac{1}{2}(r_2 + r'_p)$. Now note further that this quantity is always larger than either r'_p or $\frac{1}{2}(r_2 + r'_q)$. Thus, since we are seeking the minimum radius, the solution stated in the theorem is complete. The one exceptional case occurs where $r'_p = r'_q$, in which a symmetric solution (i. e., ϵ and δ may be interchanged) occurs.

Note that the points determined by ϵ and δ are not necessarily an absolute 2-center, but they do in general (unless $r'_p = r_2$) give a radius less than r_2.

At this point, an example of Case 3 is in order. Consider the graph of Figure 9. Since r'_p, $r'_q \le r_2$, then we must have A, B $\le$ 4. For example, let A = 2, B = 1. Then $\{(p),(q)\}$ is a vertex 2-center as are $\{(t),(q)\}$ and $\{(p),(s)\}$; $r_2 = 4$. Since $r'_p = 2 \le \frac{1}{2}(r_2 + r'_q) = 2\frac{1}{2}$, the 2-center giving the minimum radius occurs with

$$\delta = \frac{1}{2}(r_2 - r'_q) = 1\frac{1}{2},$$

$$\epsilon \le \frac{1}{2}(r_2 + r'_q) - r'_p = \frac{1}{2},$$

and

$$r = \frac{1}{2}(r_2 + r'_q) = 2\frac{1}{2}.$$

Now let A = 3, B = 1. Since $r'_p = 3 \ge \frac{1}{2}(r_2 + r'_q) = 2\frac{1}{2}$. The 2-center giving the minimum radius occurs at

$$\epsilon = 0,$$

and any

$$r_2 - r'_p \le \delta \le r'_p - r'_q$$

$$1 \le \delta \le 2,$$

with

$$r = r'_p = 3.$$

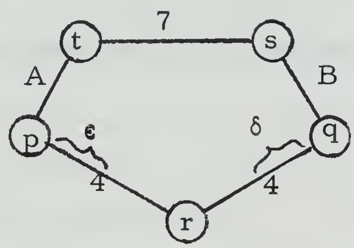

Figure 9. Example graph

Case 4. One arc equal to r_2, the other greater than r_2.

This case is in a sense an extension of Case 3 above. We will let $d(p, r) = r_2$ and $d(q, r) = R \geq r_2$. In searching for an absolute 2-center, one can occur only if we can select a point less than r_2 from (r). This can occur in two ways, either by selecting a point on (p, r) or on (q, r). In Case 3, it was unnecessary to distinguish between the cases since for that case the only difference between (p) and (q) was one of labelling. In this case, the distinction must be made.

Define ϵ to be a distance along (p, r) measured from (p), and δ to be a distance along (q, r) measured from (q).

Theorem 6. For the conditions of this case, the minimum radius obtainable for two points on (p, r) and (q, r) for this case is

$$r = \min \{ \max[\frac{1}{2}(r_2 + r'_p), r'_q], \max[\frac{1}{2}(r'_q + R), r'_p] \} .$$

If $r = \frac{1}{2}(r_2 + r'_p)$, then the points giving this radius are located at

$$\epsilon = \frac{1}{2}(r_2 - r'_p)$$

and any

$$\delta \leq \frac{1}{2}(r_2 + r'_p) - r'_q .$$

If $r = \frac{1}{2}(r'_q + R)$, then the points giving this radius are located at

$$\delta = \frac{1}{2}(R - r'_q)$$

and any

$$\epsilon \leq \frac{1}{2}(R + r'_q) - r'_p .$$

If $r = r'_p$, then the points giving this radius are located at

$$\epsilon = 0$$

and any

$$R - r'_p \leq \delta \leq r'_p - r'_q .$$

If $r = r'_q$, then the points giving this radius are located at

$$\delta = 0$$

and any

$$r_2 - r'_q \le \varepsilon \le r'_q - r'_p .$$

Proof: Recall that earlier we defined V_p as the set of vertices nearest to (p), with some distance from (p) to the most distant member of V_p. Similarly, V_q was the set of vertices nearest to (q), with some distance from (q) to the most distant member of V_q. Further, it was shown that the radius associated with {(p), (q)} was the maximum of these distances. A similar argument will be followed for points along (p, r) and (q, r).

In order to improve on r_2 (which in effect serves as an upper bound), one member of a 2-center must be closer to (r) than r_2. First consider points on (p, r). As a point moves closer to (r) on (p, r), the distance to the most distant member of V_p is increased. The minimum value this distance can be occurs where

$$r'_p + \varepsilon = r_2 - \varepsilon$$

$$\varepsilon = \frac{1}{2}(r_2 - r'_p) .$$

The distance associated from this point to (r) and the most distant member of V_p is $\frac{1}{2}(r_2 + r'_p)$. Since this point has been selected less than r_2 from (r), then the other member of the 2-center may be located on (q, r) as near as possible to the most distant member of V_q. This occurs at (q), with a distance of r'_q. It is important to remember that we are concentrating on points on (p, r) and (q, r) as possible candidates for the absolute 2-center. It may be true that we could do better than r'_q by moving onto some other arc connected to (q), but this would require considerable further study.

Summarizing the discussion thus far, a pair of points has been located which will give a radius less than r_2. The radius associated with these points will be the larger of the two distances, i. e.,

$$\max\left[\frac{1}{2}(r_2 + r'_p),\ r'_q\right] .$$

If the first term is the larger of the two, then one member of the 2-center must be located precisely at a distance

$$\epsilon = \frac{1}{2}(r_2 - r'_p),$$

in order to give this radius. The other member is free to move, so long as

$$r'_q + \delta \le \frac{1}{2}(r_2 + r'_p)$$

$$\delta \le \frac{1}{2}(r_2 + r'_p) - r'_q .$$

If, on the other hand, r'_q is the larger of the two, then one member of the 2-center must be located at (q), i.e., $\delta = 0$, in order to give this radius. The other member of the 2-center is free to move, so long as it remains within a distance of r'_q of both (r) and the most distant member of V_p. Thus

$$r_2 - r'_q \le \epsilon \le r'_q - r'_p .$$

Comparable arguments may be made examining points on (q, r). Hence, there will always be two solutions available; we would normally be interested in the one giving the minimum radius.

Note that if $R = r_2$, and (p) is selected such that $r'_p \ge r'_q$, then the results of this case reduce to the results of Case 3.

The graph of Figure 10 will serve to exemplify Case 4. Initially, let $A = 3$, $B = 3\frac{1}{4}$. The set $\{(p),(q)\}$ is a vertex 2-center (as is $\{(p),(s)\}$) with $r_2 = 4$. For the A and B given, $r = \min[\max(3\frac{1}{2}, 3\frac{1}{4}), \max(4\frac{1}{8}, 3)] = 3\frac{1}{2}$. The 2-center with the minimum

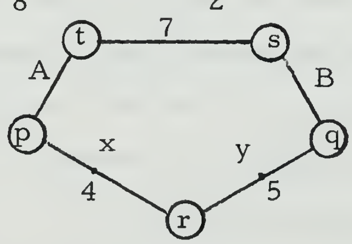

Figure 10. Example graph

radius consists of the point x located such that

$$\epsilon = \frac{1}{2}(r_2 - r'_p) = \frac{1}{2}$$

with

$$r = \frac{1}{2}(r_2 + r'_p) = 3\frac{1}{2} ,$$

and any point y so located that

$$\delta \le \frac{1}{2}(r_2 + r'_p) - r'_q = \frac{1}{4} .$$

Now let $A = 3\frac{1}{2}$, $B = 2\frac{1}{4}$, hence

$$r = \min\left[\max(3\frac{3}{4}, 2\frac{1}{4}), \max(3\frac{5}{8}, 3\frac{1}{2})\right] = 3\frac{5}{8} .$$

Thus,

$$\delta = \frac{1}{2}(R - r'_q) = 1\frac{3}{8}$$

with

$$r = \frac{1}{2}(R + r'_q) = 3\frac{5}{8} ,$$

and any

$$\epsilon \le \frac{1}{2}(R + r'_q) - r'_p = \frac{1}{8} .$$

Now let $A = 3\frac{1}{2}$, $B = 1$,

$$r = \min\left[\max(3\frac{3}{4}, 1), \max(3, 3\frac{1}{2})\right] = 3\frac{1}{2} .$$

Now

$$r = r'_p = 3\frac{1}{2}$$

$$\epsilon = 0 ,$$

and any δ

$$R - r'_p \le \delta \le r'_p - r'_q$$

$$1\frac{1}{2} \le \delta \le 2\frac{1}{2} .$$

Finally, let $A = 2$, $B = 3\frac{1}{2}$

$$r = \min\left[\max(3, 3\frac{1}{2}), \max(4\frac{1}{4}, 2)\right] = 3\frac{1}{2} .$$

For this case

$$r = r'_q = 3\frac{1}{2} ,$$

$$\delta = 0 ,$$

and any

$$r_2 - r'_q \le \varepsilon \le r'_q - r'_p$$

$$\frac{1}{2} \le \varepsilon \le 1\,\frac{1}{2} \ .$$

Case 5: One arc greater than r_2, the other less than r_2. The statement of this condition implies the existence of a vertex, call it (t), such that $d(V_2^*, t) = r_2$, where $V_2^* = \{(p), (q)\}$. The following theorem will be proved for this case.

Theorem 7. For the conditions of this case, there exists no point x on either arc such that the sets $\{(p), x\}$ or $\{x, (q)\}$ is an absolute 2-center.

Proof: Assume $d(p, r) \ge r_2$ and $d(q, r) < r_2$. One of two situations must exist. Either $d(p, t) \ge r_2$ and $d(q, t) = r_2$, or $d(p, t) = r_2$ and $d(q, t) \ge r_2$. It is not important to distinguish between them; note, however, that the distance from (p) or (q) to (t) can never be less than r_2.

Consider first a point x on the arc (q, r), the shorter of the two arcs. For this point,

$$d(x, t) = d(t, q) + \delta$$

$$d(t, q) \ge r_2$$

$$\therefore d(x, t) > r_2 \ .$$

Thus, no x located on this arc improves on r_2.

Next, consider a point x on the arc (p, r). For this point,

$$d(x, t) = \min [\, d(p, t) + \varepsilon,\ d(q, t) + d(q, r) + d(p, r) - \varepsilon \,].$$

If

$$d(x, t) = d(p, t) + \varepsilon$$

$$d(p, t) \ge r_2$$

$$\therefore d(x, t) > r_2 \ .$$

If

$$d(x, t) = d(q, t) + d(q, r) + d(p, r) - \varepsilon$$

$$d(q, t) \ge r_2$$

$$\varepsilon \le d(p, r)$$

$$\therefore d(x, t) > r_2 \ .$$

Thus no x on this arc will improve on r_2 , proving the theorem.

Note that a result of this proof is that Case 5 may be combined with Case 2.

Summarizing the results when the minimum-length path connecting the members of the vertex 2-center has one intermediate vertex with no other connectivity, it has been shown that if one of the arcs is less than r_2 , then no points on the arcs may be paired with either of the 2-center vertices to form an absolute 2-center. In all other cases when one or both of the arcs are equal to r_2 or one is equal to r_2 and the other is greater, such points will exist.

Notice that the statement is not that an absolute 2-center cannot have a member on the arcs discussed, but that no point on these arcs when paired with (p) or (q) will form an absolute 2-center. Figure 11 shows an example where a point on the arc (q, r) (in this case greater than r_2) is paired with another point to yield an absolute 2-center. For this example, $r_2 = 5$, $r_2^* = 4\frac{1}{2}$.

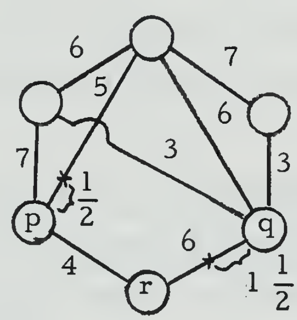

Figure 11. Example graph

CHAPTER 4

LOCATION OF ABSOLUTE 2-CENTERS

In the preceding chapters, absolute 2-centers have been defined and their existence shown. In Chapter 3, it was shown that certain arcs can either be eliminated from consideration, or can be shown to contain 2-centers which improve on the vertex 2-center immediately. In this chapter a solution technique for locating the best 2-centers on arcs incident to vertex 2-centers will be set forth.

A word of caution is in order. The solution technique as exposited here will find absolute 2-centers only if they exist on arcs incident to the vertex 2-center. It will be shown later in the chapter that absolute 2-centers can be located elsewhere. Hence it is somewhat inaccurate to refer to the points determined by the solution technique as "absolute" 2-centers, since they cannot be shown to always yield the minimum radius over the entire graph.

<u>Solution Technique</u>

1. Determine all the vertex 2-centers. This can be accomplished by scanning the D-matrix for all $\binom{n}{2}$ combinations, as explained in Chapter 3.

2. Examine the path connecting the members of any vertex 2-center. If it is a single arc, it may be eliminated from consideration in step 3. If the path contains a single vertex with no other connectivity and is the shortest path, then the arcs on this path may be eliminated from consideration in step 3. It may seem strange to entirely eliminate these arcs from consideration in step 3, since the example of Figure 11 in the preceding chapter showed that these arcs may contain members of an absolute 2-center. However, the min-distance plot procedure of step 3 is such that these solutions will be reached during examination of other arcs.

Listed below is a summary of the results for the special cases discussed in Chapter 3. In this summary, $\{(p),(q)\}$ is a vertex

2-center, (r) is a single vertex in the shortest path connecting (p) and (q), (r) having no other connectivity, ϵ is a distance measured from (p) along (p, r), and δ is a distance measured from (q) along (q, r).

I. If $d(q,r) < r_2$, then for any value of $d(p,r)$ (Cases 1, 2 and 5), no improvement on r_2 is possible.

II. If $d(p,r) = r_2$, and $d(q,r) = R \geq r_2$ (Cases 3 and 4), then

$$r = \min\{\max[\tfrac{1}{2}(r_2 + r'_p), r'_q], \max[\tfrac{1}{2}(r'_q + R), r'_p]\}, \text{ and if}$$

(a) $r = \frac{1}{2}(r_2 + r'_p)$ then $\epsilon = \frac{1}{2}(r_2 - r'_p)$,

$$0 \leq \delta \leq \tfrac{1}{2}(r_2 + r'_p) - r'_q .$$

(b) $r = \frac{1}{2}(r'_q + R)$ then $\delta = \frac{1}{2}(R - r'_q)$,

$$0 \leq \epsilon \leq \tfrac{1}{2}(R + r'_q) - r'_p .$$

(c) $r = r'_p$ then $\epsilon = 0$,

$$R - r'_p \leq \delta \leq r'_p - r'_q .$$

(d) $r = r'_q$ then $\delta = 0$,

$$r_2 - r'_q \leq \epsilon \leq r'_q - r'_p .$$

3. For all arcs incident to the members of the vertex 2-center not eliminated in step 2, make min-distance plots for absolute 2-centers as described below.

Min-distance Plots

The purpose of the min-distance plot is to determine the radius of the graph for a set of two points consisting of one member of the vertex 2-center and a point on an arc. In turn, these plots lead to determination of the radius of two points located along arcs. Specifically, if $\{(p), (q)\}$ is a vertex 2-center, then sets $\{(p), x\}$ will be examined, where x is located on arcs incident to (q). Having made the min-distance plots, we may examine them for points where the radius is less than r_2.

A simple example will serve to illustrate the plotting technique. Consider the graph of Figure 12.

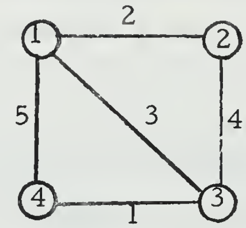

Figure 12. Example graph

There are four vertex 2-centers: $\{(1),(3)\}$, $\{(1),(4)\}$, $\{(2),(3)\}$, $\{(2),(4)\}$, all with $r_2 = 2$. For the min-distance plot, consider $\{(1),(4)\}$. Specifically, the example, Figure 13, considers the set $\{(4),x\}$, x located on the arc $(1,2)$. A plot is made of the minimum distance to each vertex as the point x moves along the arc. The abscissa of the plot is the distance from (1) to (2). The ordinate is the minimum distance to each vertex. Thus, there is a line plotted for each vertex. For example, the distance to vertex 1 (the line marked d_1) is zero when x is at vertex 1, then increases as x is moved toward vertex 2, until (2) is reached. The distance to (3) remains constant at one, the distance from (4) to (3).

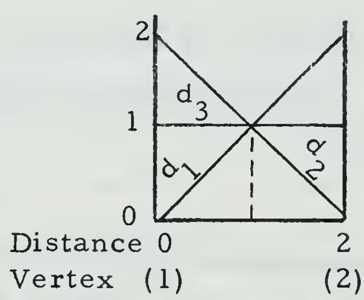

Figure 13. Example min-distance plot

Note in Figure 13 that the lines intersect where x is one unit from (1). The ordinate value at this point is one. Thus the set $\{(4),x\}$ with x located one unit from (1) on the arc $(1,2)$ yields a radius of one.

Similar plots must be made for both members of the vertex 2-center and arcs not previously eliminated (by Theorem 3, the arc (4, 1) has been eliminated). Next, examine the plot of x along (4, 3), with (1) as the other member of the 2-center. Note that the distance to (2) is constant at two. Since neither d_3 nor d_4 exceed one, the radius for this case is determined by d_2. Lowering the

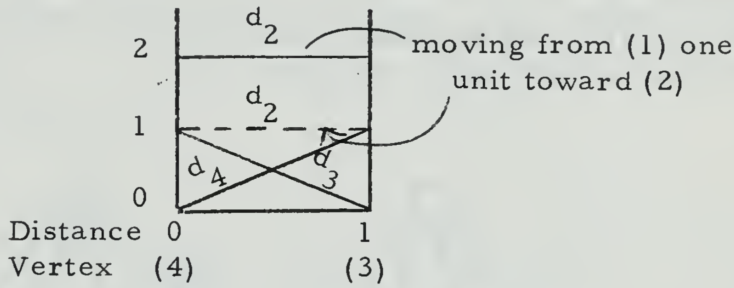

Figure 14. Example of adjusting min-distance plot

d_2 line will reduce the radius. Examining the graph, we see that we may move from (1) toward (2) one unit without affecting d_3 or d_4, as shown by the dotted line in Figure 14. This leads to the same solution reached in the preceding paragraph, but with the added information that x may be located anywhere on (4, 3) and still yield a radius of one. By examining the effect of moving off the fixed vertex, solutions on arcs previously eliminated in the algorithm will be discovered.

After plots have been made for all eligible arcs and vertices, all the plots are compared and the points yielding the minimum radius will be the most centrally located points to be found on arcs incident to a vertex 2-center.

Summarizing the procedure for the min-distance plots: first plot the radius as a function of a fixed vertex and a moving point. Then examine the largest radius and determine if it can be reduced by moving off the fixed vertex. After so adjusting all plots such that they yield a minimum 2-center, the minimum over all such points is the minimum radius obtainable for two points on arcs incident to a vertex 2-center, and may be the absolute 2-center.

It is intuitively appealing to hypothesize that absolute 2-centers should occur "near" vertex 2-centers, or at least on arcs incident to them. It does happen quite often; however, one counterexample will serve to show that it does not always happen.

For the graph of Figure 15, $\{(1),(2)\}$ is the vertex 2-center with $r_2 = 10$. The absolute 2-center occurs at the points marked

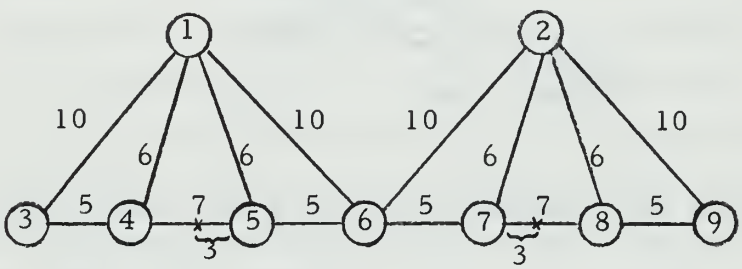

Figure 15. Example of graph with absolute 2-center not on arcs incident to the vertex 2-center

with $r_2^* = 9$. The graph has some interesting properties, e.g., the vertex 2-center connects directly to all vertices and no other pair of vertices has this property. A fruitful area for further investigation might be to determine what properties a graph must have for the absolute 2-center not to be located on arcs incident to the vertex 2-center. It may be that the counterexample comes from a very restricted class of graphs.

One example will serve to illustrate several aspects of the solution procedure. Consider the graph of Figure 16. The associated D-matrix, where $D = \{d_{ij}\}$, d_{ij} = the minimum distance from vertex i to vertex j, is

$$D = \begin{bmatrix} 0 & 2 & 5 & 3 & 5 & 4 \\ 2 & 0 & 3 & 4 & 5 & 6 \\ 5 & 3 & 0 & 3 & 2 & 4 \\ 3 & 4 & 3 & 0 & 5 & 7 \\ 5 & 5 & 2 & 5 & 0 & 2 \\ 4 & 6 & 4 & 7 & 2 & 0 \end{bmatrix} .$$

There are $\binom{6}{2}$ or 15 possible vertex 2-centers. The vertex 2-center is $\{(1),(5)\}$, with $r_2 = 3$. The method for determining r_2 was shown in detail in Chapter 2.

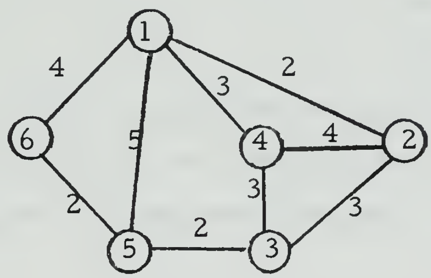

Figure 16. Example graph

The next step is to eliminate those arcs where it can be determined a priori that no member of an absolute 2-center is located. By Theorem 3, arc (1, 5) need not be considered. Likewise, by Theorem 7, arcs (1, 6) and (5, 6) are eliminated. By those steps, arcs incident to the vertex 2-center requiring consideration are reduced from seven to three.

The next step is to construct the min-distance plots. An analysis of arc (1, 4) with (5) as the other member yields Figure 17. There would be six other min-distance plots to be made but Figure 17 is sufficient to give the answer. The absolute 2-center is vertex (5) and a point located $\frac{1}{2}$ unit from (1) on arc (1, 4) resulting in $r_2^* = 2\frac{1}{2}$.

The same basic technique could be used on other arcs to locate the absolute 2-center on any arc of the graph. All vertices could

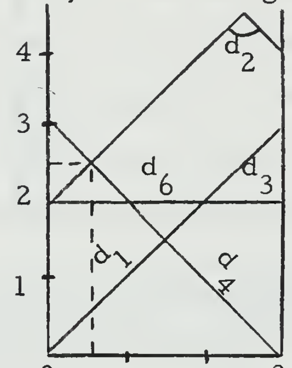

Figure 17. Min-distance plot for example

be taken pairwise, and min-distance plots made on all arcs. The amount of work involved, even for a small graph, seems prohibitive. Without further investigation leading to additional reductions in the number of arcs to be considered, the solution technique of this paper is probably not a very efficient method of locating other absolute 2-centers.

CHAPTER 5

SUMMARY

Graph theory can be usefully applied to the problem of optimal location of emergency facilities. Previous work by Hakimi[2, 3] shows how to locate the absolute center of a graph, the point, not necessarily at a vertex, which is more centrally located than any other point. Chapter 2 of this paper contains a brief account of Hakimi's work and an illustrative example.

In Chapter 3, the concept of a graph center and absolute center is extended to a vertex multi-center and absolute multi-center. The vertex multi-center is found by scanning all $\binom{n}{p}$ sets of p vertices for the set yielding the minimum radius. Several theorems are proved which specify the conditions for which certain combinations of arcs and vertices will always contain a 2-center which improves on the vertex 2-center and others will never contain an absolute 2-center. A single arc connecting the members of a vertex 2-center contains no point which may be paired with a member of the vertex 2-center to form an absolute 2-center. This arc may, however, have an interval which contains points which may be paired with a member of the vertex 2-center to yield a radius equal to the radius of the graph r_2. This same arc contains no pair of points which form an absolute 2-center.

The case where the shortest path connecting the members of the vertex 2-center contains a single vertex with no other connectivity is examined next. If both the arcs are equal to r_2 in length, or if one arc is equal to r_2 in length, the other greater than r_2, there will exist a 2-center on the shortest connecting path which will have a radius less than r_2. If one arc is less than r_2 in length, then no matter what the length of the other arc, no point will exist on either arc which may be paired with either member of the vertex 2-center to form an absolute 2-center. Both arcs cannot be greater than r_2.

A solution technique for finding the best 2-centers on arcs incident to the vertex 2-center is stated in Chapter 5. By plotting the radius of a pair of points consisting of one member of the vertex 2-center and a moving point on an arc incident to the other member, points constituting the most centrally located 2-center may be found.

Theoretically, the solution technique could be used to locate absolute 2-centers anywhere on the graph. The amount of work involved would probably be prohibitive, so the technique must be considered practically inapplicable in these cases.

There are many questions involving multi-center location still to be resolved. An area of immediate interest is to determine under what conditions absolute 2-centers may be located on arcs not incident to a vertex 2-center. The area of multiple 2-centers requires considerable investigation to determine if there is a logical order to examining them. This leads to consideration of "interlocking" 2-centers, where one vertex is a member of more than one vertex 2-center. It might be of interest to examine directed graphs, although it would appear that little could be said in general. Yet another extension would be to add weighting factors to vertices. This might be appropriate in the fire station location problem if the importance of a prompt response to an alarm varies from location to location.

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13. ABSTRACT

This study analyzes the location of vertex and absolute 2-centers of an undirected graph. Under certain assumptions, these locations would be useful for determining the optimal positioning of emergency facilities such as fire stations. Vertex and absolute multi-centers are defined, and a procedure for locating the vertex multi-centers is given. It is shown that certain combinations of arcs and vertices never contain absolute 2-centers, while certain others will always contain a 2-center which is more centrally located than the vertex 2-center. Although no algorithm was found for determining the absolute 2-center of a graph, an algorithm is presented for finding the best 2-centers which exist on arcs incident to the vertex 2-center.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Optimal Location						
Graph Theory						
Network Theory						
Emergency Facilities Location						

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